Phys 410
Spring 2013
Lecture #14 Summary
22 February, 2013

The objective is to learn how to write down Newton's second law so that it works in a non-inertial reference frame. We considered first the case of a reference frame undergoing constant linear acceleration $\vec{A}$. By comparing a description of the motion of an object as seen from an inertial reference frame to that same object seen from a non-inertial reference frame, we concluded that Newton's second law in the non-inertial reference frame must be written as $m\ddot{\vec{r}} = \vec{F}_{net} - m\vec{A}$. The "inertial force" $\vec{F}_{inertial} = -m\vec{A}$ must be added to the net force to make the equation of motion work in the non-inertial frame. We experience this inertial force as a backwards force when sitting in an aircraft that is accelerating down the runway for takeoff.

Making Newton's second law work in a rotating reference frame is more of a challenge. Consider a rigid body undergoing pure rotational motion on an axis through a point inside the object. A rigid body is one in which the distances between the particles do not change during the motion. Rotation is specified by an axis of rotation $\hat{u}$, and a rate ω. We shall assume that both the axis and the rate of rotation are fixed as a function of time. We found that the linear velocity of a particle at location $\vec{r}$ inside the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$.

We then calculated the relationship between the time-derivative of a vector $\vec{Q}$ as seen in an inertial reference frame *S₀*, to the derivative of the same vector seen in the rotating reference frame *S*. We assume that the two reference frames have the same origin, but frame *S* is rotating about an arbitrary axis $\widehat{\Omega}$ through the origin at a rate Ω. The time-derivatives are related as $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$. This equation says that the time derivative of the vector as witnessed in the inertial reference frame consists of any change in its magnitude or direction as seen in the non-inertial reference frame, plus the change brought about by the fact that the vector $\vec{Q}$ is embedded in a rotating rigid body.

Newton's second law can now be written for an observer in a rotating reference frame as $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$. The two "inertial forces" on the right are called the Coriolis force and the centrifugal force, respectively.

We considered the centrifugal force for a stationary observer on the surface of the earth. This force has a direction that is directly away from the axis of rotation of the earth and can be written as $\vec{F}_{CF} = m\Omega^2 r \sin\theta\, \hat{\rho}$, where r is the distance from the center of the earth, θ is the polar angle of the location on the surface (also known as the co-latitude) and $\hat{\rho}$ is the radial unit vector from cylindrical coordinates. This force has a maximum magnitude near the equator, but goes to zero at the poles. The centrifugal force modifies the free-fall direction. It creates a new effective gravitational acceleration vector of $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$, where $\vec{g}_0$ is the bare Newtonian gravity acceleration vector that points directly to the center of the earth, and R is the radius of the earth. The radial component of this vector is $g_{rad} = g_0 - \Omega^2 R \sin^2\theta$, showing that things weigh a bit less at the equator than at the north/south pole. The effect is small, only about 0.3%. The tangential component of $\vec{g}$ is $g_{tang} = \Omega^2 R \sin\theta \cos\theta$, with a maximum value at 45^{o} latitude. This component produces a 0.1^{o} tilt of $\vec{g}$ with respect to the direction of $\vec{g}_0$.